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An Identity Involving Partitional Generalized Binomial Coefficients

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Define coefficients $\binom{\lambda}{\kappa}$ by $C_\lambda(I_p + Z)/C_\lambda(I_p) = \sum_{k=0}^{l} \sum_{\kappa\in\mathscr{P}_k} \binom{\lambda}{\kappa} C_\kappa(Z)/C_\kappa(I_p)$, where the C_λ's are zonal polynomials in p by p matrices. It is shown that $C_\kappa(Z)\operatorname{etr}(Z)/k! = \sum_{l=k}^{\infty} \sum_{\lambda\in\mathscr{P}_l} \binom{\lambda}{\kappa} C_\lambda(Z)/l!$. This identity is extended to analogous identities involving generalized Laguerre, Hermite, and other polynomials. Explicit expressions are given for all $\binom{\lambda}{\kappa}$, $\kappa\in\mathscr{P}_k$, $k \leqslant 3$. Several identities involving the $\binom{\lambda}{\kappa}$'s are derived. These are used to derive explicit expressions for coefficients of $C_\lambda(Z)/l!$ in expansions of $P(Z)\operatorname{etr}(Z)/k!$ for all monomials $P(Z)$ in $s_j = \operatorname{tr} Z^j$ of degree $k \leqslant 5$.

1. INTRODUCTION

Constantine [2] defined a generalized "binomial" type coefficient $\binom{\lambda}{\kappa}$ (his notation was $a_{\lambda,\kappa}$) by the expansion

$$C_\lambda(I_p + Z)/C_\lambda(I_p) = \sum_{k=0}^{l} \sum_{\kappa\in\mathscr{P}_k} \binom{\lambda}{\kappa} C_\kappa(Z)/C_\kappa(I_p) \tag{1.1}$$

where $\mathscr{P}_k$ is the set of all partitions $\kappa = (k_1, k_2, \ldots, k_p)$, $k_{i-1} \geqslant k_i$, $\sum_{i=1}^{p} k_i = k$, $\lambda\in\mathscr{P}_l$, and the $C_\kappa(Z)$'s are zonal polynomials as defined by James [6]. The principal result derived below is the identity

$$\sum_{l=k}^{\infty} \sum_{\lambda\in\mathscr{P}_l} \binom{\lambda}{\kappa} C_\lambda(Z)/l! = C_\kappa(Z)\operatorname{etr}(Z)/k! \tag{1.2}$$

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where $\operatorname{etr}(Z) = \exp(\operatorname{tr} Z)$. This identity is extended to analogous sums involving generalized Laguerre polynomials, generalized Hermite polynomials, and other polynomials. In addition it is used to derive several recurrence relations among the $\binom{\lambda}{\kappa}$'s.

By analogy with (1.2) (although not with (1.1)) it is convenient to introduce the following notation.

DEFINITION. Let $P(Z)$ be a homogeneous symmetric polynomial of degree k in Z. Then the coefficients $\binom{\lambda}{P}$ are defined by

$$P(Z)\operatorname{etr}(Z)/k! = \sum_{l=k}^{\infty} \sum_{\lambda \in \mathscr{P}_l} \binom{\lambda}{P} C_\lambda(Z)/l!\,. \tag{1.3}$$

Expressions are presented for $\binom{\lambda}{P}$ for all monomials P in $s_j = \operatorname{tr} Z^j$ of degree $r \leqslant 5$.

2. The Principal Result

James [6] defined the function

$${}_0F_0^{(p)}(Z, T) = V_p^{-1} \int_{O(p)} \operatorname{etr}(ZH'TH)(dH) \tag{2.1}$$

where $O(p)$ is the group of orthogonal $p \times p$ matrices with invariant (Haar) measure (dH) with total content V_p. Using the identity

$$V_p^{-1} \int_{O(p)} C_\kappa(ZH'TH)(dH) = C_\kappa(Z)\, C_\kappa(T)/C_\kappa(I_p) \tag{2.2}$$

he derives the expansion

$${}_0F_0^{(p)}(Z, T) = \sum_{k=0}^{\infty} (1/k!) \sum_{\kappa \in \mathscr{P}_k} C_\kappa(Z)\, C_\kappa(T)/C_\kappa(I_p). \tag{2.3}$$

Thus ${}_0F_0^{(p)}(Z, T)$ can be considered to be a generating function for the zonal polynomials $C_\kappa(Z)$.

LEMMA 1. *Let z and t be arbitrary scalars. Then*

$$\operatorname{etr}(-ZT)\,{}_0F_0^{(p)}(Z, T) = \operatorname{etr}(-(Z + zI_p)(T + tI_p))\,{}_0F_0^{(p)}(Z + zI_p, T + tI_p).$$

Proof. Consider the exponent in the integrand of (2.1).

$$\begin{aligned}\operatorname{tr} ZH'TH &= \operatorname{tr}(Z + zI_p)\, H'(T + tI_p)\, H - z \operatorname{tr}(T) - t \operatorname{tr}(Z) - pzt \\ &= \operatorname{tr}(Z + zI_p)\, H'(T + tI_p)\, H + \operatorname{tr}(ZT) - \operatorname{tr}(Z + zI_p)(T + tI_p).\end{aligned}$$

The result now is immediate. ∎

The main result is the following.

THEOREM 1. *Let the generalized binomial coefficients be defined by* (1.1), *and let* $\kappa \in \mathscr{P}_k$. *Then*

$$\sum_{l=k}^{\infty} \sum_{\lambda \in \mathscr{P}_l} \binom{\lambda}{\kappa} C_\lambda(Z)/l! = C_\kappa(Z) \operatorname{etr}(Z)/k!. \tag{2.4}$$

Proof. By (2.3) a generating function for $C_\kappa(Z) \operatorname{etr}(Z)$ can be taken to be

$$\operatorname{etr}(Z)\, {}_0F_0^{(p)}(Z, T) = \sum_{k=0}^{\infty} (1/k!) \sum_{\kappa \in \mathscr{P}_k} [C_\kappa(Z) \operatorname{etr}(Z)]\, C_\kappa(T)/C_\kappa(I_p).$$

Using Lemma 1 with $z = 0$ and $t = 1$ we also have

$$\begin{aligned}\operatorname{etr}(Z)\, {}_0F_0^{(p)}(Z, T) &= {}_0F_0^{(p)}(Z, I_p + T) \\ &= \sum_{l=0}^{\infty} (1/l!) \sum_{\lambda \in \mathscr{P}_l} C_\lambda(Z)\, C_\lambda(I_p + T)/C_\lambda(I_p).\end{aligned}$$

From (1.1) we thus have

$$\begin{aligned}\operatorname{etr}(Z)\, {}_0F_0^{(p)}(Z, T) &= \sum_{l=0}^{\infty} (1/l!) \sum_{\lambda \in \mathscr{P}_l} C_\lambda(Z) \sum_{k=0}^{l} \sum_{\kappa \in \mathscr{P}_k} \binom{\lambda}{\kappa} C_\kappa(T)/C_\kappa(I_p) \\ &= \sum_{k=0}^{\infty} \sum_{\kappa \in \mathscr{P}_k} \left[\sum_{l=k}^{\infty} \sum_{\lambda \in \mathscr{P}_l} \binom{\lambda}{\kappa} C_\lambda(Z)/l! \right] C_\kappa(T)/C_\kappa(I_p).\end{aligned}$$

Comparing coefficients of $C_\kappa(T)/C_\kappa(I_p)$ the theorem is proved. ∎

Several expansions in zonal polynomials of $P(Z) \operatorname{etr}(Z)$, where $P(Z)$ is a symmetric polynomial in the latent roots of Z, have appeared in the literature [9, 3]. Since any such $P(Z)$ can be expressed in terms of zonal polynomials, the coefficients appearing in these expansions can be expressed in terms of generalized binomial coefficients. Using, for example, the summary in [3], we can thus find all $\binom{\lambda}{\kappa}$ for $\kappa \in \mathscr{P}_k$, $k \leqslant 3$. It is convenient to use a notation that differs from that which has generally been used previously.

DEFINITION. Let $\kappa = (k_1, k_2, \ldots, k_p) \in \mathscr{P}_k$. Then

$$d_0(\kappa) \equiv 1, \quad d_1(\kappa) \equiv k, \quad d_r(\kappa) \equiv \sum_{i=1}^{p} \sum_{j=1}^{k_i} (j - (1/2)(i + 1))^{r-1}, \quad r = 2, 3, \ldots. \tag{2.5}$$

Moreover, it is easily checked that Suguira and Fujikoshi's [9] $a_1(\kappa)$ and $a_2(\kappa)$ and Fujikoshi's [3] $a_3(\kappa)$ can be expressed as

$$\begin{aligned} a_1(\kappa) &= 2d_2(\kappa), \\ a_2(\kappa) &= 12d_3(\kappa) + k, \\ a_3(\kappa) &= 8d_4(\kappa) + 2d_2(\kappa). \end{aligned} \tag{2.6}$$

Conversely,

$$\begin{aligned} d_1(\kappa) &= k, \\ d_2(\kappa) &= (1/2)\, a_1(\kappa), \\ d_3(\kappa) &= (1/12)(a_2(\kappa) - k), \\ d_4(\kappa) &= (1/8)(a_3(\kappa) - a_1(\kappa)). \end{aligned} \tag{2.7}$$

Using (2.6), the series of Suguira and Fujikoshi [9] yield the results [$s_r \equiv \operatorname{tr}(Z^r)$], using the notation of (1.3),

$$\begin{aligned} \binom{\lambda}{s_1{}^k} &= \binom{l}{l-k}, \\ \binom{\lambda}{s_2} &= d_2(\lambda), \\ \binom{\lambda}{s_1 s_2} &= (1/3)(l-2)\, d_2(\lambda), \\ \binom{\lambda}{s_3} &= (1/2)\, d_3(\lambda) - (1/4)\, d_2(\lambda) - (1/4)\binom{l}{2}. \end{aligned} \tag{2.8}$$

Using the explicit expressions for low order $C_\kappa(Z)$ in [6], (2.8) and (2.4) yield the following explicit expressions for $\binom{\lambda}{\kappa}$, $\kappa \in \mathscr{P}_k$, $k \leqslant 3$ and arbitrary $\lambda \in \mathscr{P}_l$.

$$\begin{aligned} \binom{\lambda}{(0)} &= 1, \\ \binom{\lambda}{(1)} &= l, \\ \binom{\lambda}{(2)} &= (1/3)\left[\binom{l}{2} + 2d_2(\lambda)\right], \\ \binom{\lambda}{(1^2)} &= (1/3)\left[2\binom{l}{2} - 2d_2(\lambda)\right], \\ \binom{\lambda}{(3)} &= (1/15)\left[\binom{l}{3} + 2(l-2)\, d_2(\lambda) + 4d_3(\lambda) - 2\left(\binom{l}{2} + d_2(\lambda)\right)\right], \\ \binom{\lambda}{(21)} &= (1/15)\left[9\binom{l}{3} + 3(l-2)\, d_2(\lambda) - 9d_3(\lambda) + (9/2)\left(\binom{l}{2} + d_2(\lambda)\right)\right], \\ \binom{\lambda}{(1^3)} &= (1/15)\left[5\binom{l}{3} - 5(l-2)\, d_2(\lambda) + 5d_3(\lambda) - (5/2)\left(\binom{l}{2} + d_2(\lambda)\right)\right]. \end{aligned} \tag{2.9}$$

Methods for deriving and extending (2.8) using identities among the $\binom{\lambda}{\kappa}$'s are given in Section 3. The appendix contains an extension to (2.8) through degree 5 in Z.

3. Identities Involving $\binom{\lambda}{\kappa}$

Although no general formula for $\binom{\lambda}{\kappa}$ is known at this time, various identities can be derived enabling one to find expressions for all $\binom{\lambda}{\kappa}$, $\kappa \in \mathscr{P}_k$, $k \leqslant 5$.

Theorem 2. *Let $P(Z)$ be a homogeneous symmetric polynomial of degree k in Z. Then, if $s_1 = \operatorname{tr} Z$,*

$$s_1{}^r P(Z) = \binom{k+r}{r}^{-1} \sum_{\sigma \in \mathscr{P}_{k+r}} \binom{\sigma}{P} C_\sigma(Z) \tag{3.1}$$

and, if $\sigma \in \mathscr{P}_s$, $s \geqslant r + k$,

$$\binom{\sigma}{s_1{}^r P} = \binom{k+r}{r}^{-1} \binom{s-k}{r} \binom{\sigma}{P}. \tag{3.2}$$

Furthermore, for $\sigma \in \mathscr{P}_s$, $k \leqslant l \leqslant s$,

$$\sum_{\lambda \in \mathscr{P}_l} \binom{\sigma}{\lambda}\binom{\lambda}{P} = \binom{s-k}{l-k}\binom{\sigma}{P}. \tag{3.3}$$

Proof. Expanding $\operatorname{etr} Z = \sum_{r=0}^{\infty} s_1{}^r/r!$ on the l.h.s. of (1.3) and matching terms of like degree yields (3.1). Expanding $s_1{}^r P(Z) \operatorname{etr} Z/(k+r)!$ similarly and substituting (3.1) yields (3.2). Also we have

$$s_1{}^r P(Z) \operatorname{etr} Z/(k+r)! = (k!/(k+r)!) \sum_{l=k}^{\infty} \sum_{\lambda \in \mathscr{P}_l} \binom{\lambda}{P} s_1{}^r C_\lambda(Z)/l!.$$

But by (3.1) applied to $P(Z) = C_\lambda(Z)$ and Theorem 1, this is

$$(k!/(k+r)!) \sum_{l=k}^{\infty} \sum_{\lambda \in \mathscr{P}_l} \binom{\lambda}{P} \left[\binom{l+r}{r}^{-1} \sum_{\sigma \in \mathscr{P}_{k+r}} \binom{\sigma}{\lambda} C_\sigma(Z)\right] \Big/ l!$$

$$= \binom{k+r}{r}^{-1} \sum_{s=k+r}^{\infty} \sum_{\sigma \in \mathscr{P}_s} \left[\sum_{\lambda \in \mathscr{P}_{s-r}} \binom{\sigma}{\lambda}\binom{\lambda}{P}\right] C_\sigma(Z)/s!.$$

This yields a second expression for $\binom{\sigma}{s_1{}^r P}$ thus establishing (3.3). ∎

Remark. Given expressions for $C_\kappa(Z)$ and all $C_\lambda(Z)$, $\lambda \in \mathscr{P}_l$, in terms of monomials in $s_r = \operatorname{tr} Z^r$, (3.2) applied to $P(Z) = C_\kappa(Z)$ provides a simple means of computing $\binom{\lambda}{\kappa}$ for small values of k and l. Its application is somewhat simplified by the following.

LEMMA 2. *Let* $\lambda = (l_1, ..., l_p) \in \mathscr{P}_l$, $\kappa = (k_1, ..., k_p) \in \mathscr{P}_k$. *Then*

$$\binom{\lambda}{\kappa} = 0 \qquad \text{if any} \quad l_i < k_i, \quad i = 1, ..., p. \tag{3.4}$$

Proof. By (1.1),

$$C_\lambda(tI_p + Z)/C_\lambda(I_p) = \sum_{r=0}^{l} t^r \sum_{\kappa \in \mathscr{P}_{l-r}} \binom{\lambda}{\kappa} C_\kappa(Z)/C_\kappa(I_p).$$

But also, by Taylor's theorem,

$$C_\lambda(tI_p + Z) = \sum_{r=0}^{l} (t^r/r!)[(d/dt)^r C_\lambda(Z + tI_p)|_{t=0}].$$

Without loss of generality we can assume that $Z = \operatorname{diag}[z_1, ..., z_p]$. Then, for any function $f(Z)$, $(d/dt) f(Z + tI_p) = \epsilon f(Z + tI_p)$, where $\epsilon = \sum_{j=1}^{p} (\partial/\partial z_j)$, as in [7]. Thus we have the identity

$$\sum_{\kappa \in \mathscr{P}_{l-r}} \binom{\lambda}{\kappa} C_\kappa(Z)/C_\kappa(I_p) = (1/r!)\, \epsilon^r C_\lambda(Z)/C_\lambda(I_p).$$

But by [7],

$$\epsilon C_\lambda(Z)/C_\lambda(I_p) = \sum_{i=1}^{p} \binom{\lambda}{\lambda^{(i)}} C_{\lambda^{(i)}}(Z)/C_{\lambda^{(i)}}(I_p),$$

where $\lambda^{(i)} = (l_1, ..., l_i - 1, l_{i+1}, ..., l_p)$, any $\lambda^{(i)}$ with $l_i - 1 < l_{i+1}$ being omitted from the sum. Thus the application of ϵ^r to $C_\lambda(Z)$ produces only $C_\kappa(Z)$, $\kappa = (k_1, ..., k_p) \in \mathscr{P}_k$ satisfying $k_i \leqslant l_i$, $i = 1, ..., p$. ∎

We note that (3.4) can be verified empirically up to $l = 8$ in the Table of Pillai and Jouris [8].

An interesting identity can be obtained by applying (3.3) to $P(Z) = C_k(Z)$:

$$\sum_{\lambda \in \mathscr{P}_l} \binom{\lambda}{\kappa} = \binom{s-k}{l-k} \binom{\sigma}{\kappa}, \qquad k \leqslant l \leqslant s, \quad \sigma \in \mathscr{P}_s. \tag{3.5}$$

When $k = 0$ this yields

$$\sum_{\lambda \in \mathscr{P}_\lambda} \binom{\sigma}{\lambda} = \binom{s}{l}, \qquad \sigma \in \mathscr{P}_s .$$

Another set of identities can be derived as follows.

LEMMA 3. *Let $\lambda \in \mathscr{P}_l$ and let a and b be arbitrary. Then*

$$\sum_{k=0}^{l} (-1)^k \sum_{\kappa \in \mathscr{P}_k} \binom{\lambda}{\kappa} (a)_\kappa/(b)_\kappa = (b-a)_\lambda/(b)_\lambda , \tag{3.6}$$

where

$$(x)_\rho = \prod_{i=1}^{p} (x - \tfrac{1}{2}(i-1))_{r_i} ,$$

$$\rho = (r_1 ,..., r_p) \in \mathscr{P}_r , \qquad \text{and} \qquad (y)_r = y(y+1) \cdots (y+r-1).$$

Proof. Kummer's identity, extended by Herz [5], gives

$$\operatorname{etr}(Z)\,{}_1F_1(a; b; -Z) = {}_1F_1(b-a; b; Z), \tag{3.7}$$

where ${}_1F_1(a; b; Z)$ is a confluent hypergeometric function of matrix argument shown by Constantine [1] to be expressible as

$${}_1F_1(a; b; Z) = \sum_{k=0}^{\infty} \sum_{\kappa \in \mathscr{P}_k} [(a)_\kappa/(b)_\kappa]\, C_\kappa(Z)/k!.$$

Expanding both sides of (3.7) in zonal polynomials and applying Theorem 1 to the left hand side yields (3.6). ∎

Let $b = n$ and $a = n - x$. Then we obtain recurrence relations in the $\binom{\lambda}{\kappa}$ by expanding both sides of the identity

$$\sum_{k=0}^{l} (-1)^k \sum_{\kappa \in \mathscr{P}_k} \binom{\lambda}{\kappa} (x)_\kappa/(n)_\kappa = (n-x)_\lambda/(n)_\lambda \tag{3.8}$$

in powers of x and n^{-1} and matching coefficients on each side. Now, if $\lambda = (l_1 ,..., l_p)$,

$$\begin{aligned}(n-x)_\lambda/(n)_\lambda &= \exp\left\{\sum_{i=1}^{p} \sum_{j=1}^{l_i} \log(1 - x/[n + j - \tfrac{1}{2}(i+1)])\right\} \\ &= \exp\left\{- \sum_{r=1}^{\infty} (x^r/r) \sum_{i=1}^{p} \sum_{j=1}^{l_i} [n + j - \tfrac{1}{2}(i+1)]^{-r}\right\} \\ &= \exp\left\{- \sum_{r=1}^{\infty} (x^r/r) \sum_{k=0}^{\infty} [(r)_k(-1)^k/(n^{r+k}k!)]\, d_{k+1}(\lambda)\right\},\end{aligned}$$

where $d_{k+1}(\lambda)$ is defined by (2.5). After some simplification, we find

$$
\begin{aligned}
&(n-x)_\lambda/(n)_\lambda \\
&\quad = 1 + \left[\sum_{r=1}^{\infty} (-1)^r\, d_r(\lambda) n^{-r}\right] x + \sum_{s=2}^{l} (-1)^s x^s \left\{\binom{l}{s} n^{-s} - d_2(\lambda)\binom{l-1}{s-1} n^{-s-1}\right. \\
&\qquad + \left[\tfrac{1}{2}(d_2^2(\lambda) + d_3(\lambda))\binom{l-2}{s-2} + d_3(\lambda)\binom{l-2}{s-1}\right] n^{-s-2} \\
&\qquad - \left[(\tfrac{1}{6})(d_2^3(\lambda) + 3d_2(\lambda)\, d_3(\lambda) + 2d_4(\lambda))\binom{l-3}{s-3}\right. \\
&\qquad \left.\left. + (d_2(\lambda)\, d_3(\lambda) + d_4(\lambda))\binom{l-3}{s-2} + d_4(\lambda)\binom{l-3}{s-1}\right] n^{-s-3} + O(n^{-s-4})\right\}.
\end{aligned}
\tag{3.9}
$$

Define coefficients $f_r(\kappa)$ and $Q_s(\kappa)$ by

$$
(x)_\kappa = \sum_{r=0}^{k} f_{k-r}(\kappa) x^r, \qquad (n)_\kappa^{-1} = \sum_{s=0}^{\infty} Q_s(\kappa) n^{-s-k}. \tag{3.10}
$$

Then

$$
\begin{aligned}
&\sum_{k=0}^{l} (-1)^k \sum_{\kappa\in\mathscr{P}_k} \binom{\lambda}{\kappa} (x)_\kappa/(n)_\kappa \\
&\quad = \sum_{s=0}^{l} \sum_{t=0}^{\infty} \left[\sum_{k=s}^{s+t} (-1)^k \sum_{\kappa\in\mathscr{P}_k} \binom{\lambda}{\kappa} f_{k-s}(\kappa)\, Q_{t-k+s}(\kappa)\right] \cdot (x^s/n^{s+t}).
\end{aligned}
\tag{3.11}
$$

Matching coefficients of $x^s n^{-s-t}$ in (3.9) and (3.11), and observing that $Q_0(\kappa) = f_0(\kappa) = 1$, $Q_j((0)) = Q_j((1)) = 0$, $j \geqslant 1$, we obtain

$$
\sum_{\rho\in\mathscr{P}_r} f_{r-1}(\rho)\binom{\lambda}{\rho} = d_r(\lambda) - \sum_{k=1}^{r-2} (-1)^k \sum_{\rho'\in\mathscr{P}_{r-k}} Q_k(\rho') f_{r-k-1}(\rho')\binom{\lambda}{\rho'}, \tag{3.12}
$$

$$
\sum_{\rho\in\mathscr{P}_r} f_1(\rho)\binom{\lambda}{\rho} = \binom{l-1}{r-2} d_2(\lambda) + \sum_{\rho'\in\mathscr{P}_{r-1}} Q_1(\rho')\binom{\lambda}{\rho'}, \tag{3.13}
$$

$$
\begin{aligned}
\sum_{\rho\in\mathscr{P}_r} f_2(\rho)\binom{\lambda}{\rho} &= \binom{l-2}{r-3} d_3(\lambda) + \tfrac{1}{2}\binom{l-2}{r-4}[d_2^2(\lambda) + d_3(\lambda)] \\
&\quad + \sum_{\rho'\in\mathscr{P}_{r-1}} Q_1(\rho') f_1(\rho')\binom{\lambda}{\rho'} - \sum_{\rho''\in\mathscr{P}_{r-2}} Q_2(\rho'')\binom{\lambda}{\rho''},
\end{aligned}
\tag{3.14}
$$

$$
\begin{aligned}
\sum_{\rho\in\mathscr{P}_r} f_3(\rho)\binom{\lambda}{\rho} &= \binom{l-3}{r-4} d_4(\lambda) + \binom{l-3}{r-5}[d_2(\lambda)\, d_3(\lambda) + d_4(\lambda)] \\
&\quad + \tfrac{1}{6}\binom{l-3}{r-6}[d_2^3(\lambda) + 3d_3(\lambda) + 2d_4(\lambda)] \\
&\quad + \sum_{\rho'\in\mathscr{P}_{r-1}} Q_1(\rho') f_2(\rho')\binom{\lambda}{\rho'} - \sum_{\rho''\in\mathscr{P}_{r-2}} Q_2(\rho'') f_1(\rho'')\binom{\lambda}{\rho''} \\
&\quad + \sum_{\rho'''\in\mathscr{P}_{r-3}} Q_3(\rho''')\binom{\lambda}{\rho'''}.
\end{aligned}
\tag{3.15}
$$

The left-hand sides of these relations can be associated with known homogeneous symmetric polynomials as follows. By [1], if $\| Z \| < 1$,

$$[\det(I_p - Z)]^{-\alpha} = {}_1F_0(\alpha; Z) = \sum_{r=0}^{\infty} \sum_{\rho \in \mathscr{P}_r} (\alpha)_\rho C_\rho(Z)/r!. \tag{3.16}$$

The l.h.s. of (3.16) can be expressed as

$$\begin{aligned}\exp[-\alpha \log \det(I_p - Z)] &= \exp\left[\alpha \sum_{j=1}^{\infty} s_j/j\right] \\ &= \sum_{t=0}^{\infty} (\alpha^t/t!) \left[\sum_{j=1}^{\infty} s_j/j\right]^t. \end{aligned} \tag{3.17}$$

Using (3.10) the r.h.s. of (3.16) is

$$\sum_{r=0}^{\infty} \sum_{\rho \in \mathscr{P}_r} \left[\sum_{t=0}^{r} f_{r-t}(\rho)\alpha^t\right] C_\rho(Z)/r! = \sum_{t=0}^{\infty} \alpha^t \sum_{r=t}^{\infty} \sum_{\rho \in \mathscr{P}_r} f_{r-t}(\rho)\, C_\rho(Z)/r!. \tag{3.18}$$

Thus matching terms of equal degree in Z in the coefficients of α^t in (3.17) and (3.18) we obtain

$$\begin{aligned}\sum_{\rho \in \mathscr{P}_r} f_{r-t}(\rho)\, C_\rho(Z)/r! &= \left[\text{term of degree } r \text{ in } \left(\sum_{j=1}^{\infty} s_j/j\right)^t \Big/ t!\right] \\ &\equiv P_{r,t}/r!. \end{aligned} \tag{3.19}$$

Multiplying (3.19) by etr$Z$ and expanding the l.h.s. using Theorem 1 we obtain

$$\sum_{\rho \in \mathscr{P}_r} f_{r-t}(\rho) \binom{\lambda}{\rho} = \binom{\lambda}{P_{r,t}}.$$

Thus both sides of (3.12) through (3.15) are expressions for $\binom{\lambda}{P}$, P a known polynomial. The $P(Z)$ corresponding to each of these expressions are in Table 1.

TABLE 1

Polynomials Associated with $\binom{\lambda}{P}$ Given by Eqs. (3.12)–(3.15)

Eq. for $\binom{\lambda}{P}$	$P(Z)$
(3.12)	$(r-1)!\, s_r$
(3.13)	$r(r-1)\, s_1^{r-2}[s_2/2]$
(3.14)	$r(r-1)(r-2)\, s_1^{r-4}[s_1 s_3/3 + (r-3)s_2^2/8]$
(3.15)	$r(r-1)(r-2)(r-3)\, s_1^{r-6}$
	$\times\, [s_1^2 s_4/4 + (r-4)s_1 s_2 s_3/6 + (r-4)(r-5)s_2^3/48]$

By Theorem 1, if we know $\binom{\lambda}{P}$ for all the monomials $P(Z)$ in s_j of degree r, we can find all $\binom{\lambda}{P}$. By (3.2) of Theorem 2, we need only find $\binom{\lambda}{P}$ for monomials in s_j, $j \geqslant 2$. Inspection of Table 1 indicates that it can be used to find $\binom{\lambda}{P}$, for all monomials P in s_j of degree $\leqslant 5$, and hence all $\binom{\lambda}{\rho}$, $\rho \in \mathscr{P}_r$, $r \leqslant 5$. However, even were the table extended, it would be insufficient to find all $\binom{\lambda}{\rho}$, $\rho \in \mathscr{P}_6$, since the term of degree 6 in $(\sum_{j=1}^{\infty} s_j/j)^2$ is $((2s_1s_5/5) + (s_2s_4/4) + (s_3{}^2/9))$, so that s_2s_4 and $s_3{}^2$ are not "separable" by this method. The expansions associated with all monomials of degrees 4 and 5 derived from Table 1 are given in the Appendix. Explicit expressions for $\binom{\lambda}{\rho}$, $\rho \in \mathscr{P}_4$ and $\rho \in \mathscr{P}_5$ can then be found using the expressions for zonal polynomials of degrees 4 and 5 in [6].

4. Other Analogous Expansions

In this section are given several identities analogous to (2.4) involving other polynomials that can be defined by means of zonal polynomials. The proofs are straightforward and only brief indications are provided.

Theorem 3. *Let Z be such that $\| Z \| < 1$, $\kappa \in \mathscr{P}_k$ and let b be arbitrary. Then*

$$\sum_{l=k}^{\infty} \sum_{\lambda \in \mathscr{P}_l} \binom{\lambda}{\kappa} (b)_\lambda C_\lambda(Z)/l! = [\det(I_p - Z)]^{-b} (b)_\kappa C_\kappa((I_p - Z)^{-1}Z)/k!. \quad (4.1)$$

Proof. Define, as in [1]

$${}_1F_0^{(p)}(b; Z, T) = V_p^{-1} \int_{O(p)} [\det(I_p - ZH'TH)]^{-b} (dH), \quad \| Z \| < 1, \quad \| T \| < 1.$$

Then by (3.16) and (2.2)

$${}_1F_0^{(p)}(b; Z, T) = \sum_{k=0}^{\infty} (1/k!) \sum_{\kappa \in \mathscr{P}_k} (b)_\kappa C_\kappa(Z)\, C_\kappa(T)/C_\kappa(I_p).$$

The result then follows in a manner similar to Theorem 1 from the easily derivable identity

$${}_1F_0^{(p)}(b; Z, I_p + T) = [\det(I_p - Z)]^{-b}\, {}_1F_0^{(p)}(b; (I_p - Z)^{-1}Z, T). \quad \blacksquare$$

Corollary. *Let f_λ and g_κ be such that*

$$\sum_{l=0}^{\infty} \sum_{\lambda \in \mathscr{P}_l} f_\lambda C_\lambda(Z)/l! = \left[\sum_{k=0}^{\infty} \sum_{\kappa \in \mathscr{P}_k} g_\kappa C_\kappa(Z)/k! \right] \operatorname{etr}(Z). \quad (4.2)$$

Then

$$\sum_{l=0}^{\infty} \sum_{\lambda \in \mathscr{P}_l} (b)_\lambda f_\lambda C_\lambda(Z)/l! = [\det(I - Z)]^{-b} \sum_{k=0}^{\infty} \sum_{\kappa \in \mathscr{P}_k} (b)_\kappa g_\kappa C_\kappa((I - Z)^{-1}Z)/k!. \quad (4.3)$$

Proof. (4.2) and Theorem 1 imply the identity

$$f_\lambda = \sum_{k=0}^{l} \sum_{\kappa \in \mathscr{P}_k} g_\kappa \binom{\lambda}{\kappa}, \qquad \lambda \in \mathscr{P}_l .$$

When this is substituted in the l.h.s. of (4.3), (3.1) then gives the r.h.s. of (4.3). ∎

By the Corollary, Fujikoshi's [3] Eqs. (2.6)–(2.10) imply his Eqs. (2.16)–(2.20) when one expresses the $C_\kappa((I - Z)^{-1}Z)$ in terms of monomials in $\mathrm{tr}[(I - Z)^{-1}Z]^r$.

Hayakawa [4] defines a family of polynomials $P_\kappa(T, A)$ in a $p \times n$ matrix T and an $n \times n$ symmetric matrix A by

$$\mathrm{etr}(-TT')\, P_\kappa(T, A) = (-1)^k \pi^{-(1/2)pn} \int_U \mathrm{etr}(-2iTU')\, \mathrm{etr}(-UU')\, C_\kappa(UAU')\, dU, \quad (4.4)$$

where $U = [u_{ij}]$ is a real $p \times n$ matrix, and the domain of integration is over $-\infty < u_{ij} < +\infty$, all u_{ij}; $dU = \prod_{i=1}^{p} \prod_{j=1}^{n} du_{ij}$. Particular cases of $P_\kappa(T, A)$ are [4, Theorem 6, corrected]

$$P_\kappa(0, A) = (-1)^k (p/2)_\kappa C_\kappa(A), \quad (4.5)$$

$$P_\kappa(T, I_n) = H_\kappa(T), \quad (4.6)$$

where $H_\kappa(T)$ is a particular form of the generalized Hermitian polynomial defined by Herz [5] and Hayakawa [4].

THEOREM 4. *Let* $\kappa \in \mathscr{P}_k$, $\| A \| < 1$, *and let* $(I_n - A)^{1/2}$ *be a symmetric square root of* $I_n - A$. *Then*

$$\begin{aligned} &\sum_{l=k}^{\infty} \sum_{\lambda \in \mathscr{P}_l} (-1)^l \binom{\lambda}{\kappa} P_\lambda(T, A)/l! \\ &\quad = (-1)^k\, \mathrm{etr}(-(I_n - A)^{-1} A T'T) [\det(I_n - A)]^{-(1/2)p} \\ &\quad \cdot P_\kappa(T(I_n - A)^{-(1/2)}, (I_n - A)^{-(1/2)} A (I_n - A)^{-(1/2)})/k!. \qquad (4.7) \end{aligned}$$

Proof. From (4.4) the l.h.s. of (4.7) is

$$\mathrm{etr}(TT')\, \pi^{-(1/2)pn} \int_U \left\{ \mathrm{etr}(-2iTU')\, \mathrm{etr}(-UU') \sum_{l=k}^{\infty} \sum_{\lambda \in \mathscr{P}_l} \binom{\lambda}{\kappa} C_\lambda(UAU')/l! \right\} dU.$$

The result follows by an application of Theorem 1 and a change of variables to $V = U(I_n - A)^{1/2}$. ∎

THEOREM 5. *Let* $|x| < 1$, $\kappa \in \mathscr{P}_k$, *and let* T *be a* $p \times n$ *matrix. Then*

$$\sum_{l=k}^{\infty} \sum_{\lambda \in \mathscr{P}_l} (-1)^l \binom{\lambda}{\kappa} x^l H_\lambda(T)/l!$$

$$= (-1)^k \operatorname{etr}(-x(1-x)^{-1}TT')(1-x)^{-(1/2)pn} \cdot [x(1-x)^{-1}]^k H_\kappa((1-x)^{-(1/2)}T)/k!. \tag{4.8}$$

Proof. This follows directly from Theorem 4, observing that (4.4) and (4.6) imply that $x^l H_\lambda(T) = P_\lambda(T, xI_n)$. ∎

Constantine [2], following Herz [5], defines generalized Laguerre polynomials to be, for $\lambda \in \mathscr{P}_l$ and $q = 2\gamma + p + 1$,

$$L_\lambda^\gamma(Z) = \operatorname{etr}(Z)[\Gamma_p(\tfrac{1}{2}q)]^{-1} \int_{R>0} \operatorname{etr}(-R)(\det R)^\gamma \, {}_0F_1(\tfrac{1}{2}q; -RZ)\, C_\lambda(R)\, dR, \tag{4.9}$$

where the integral is over all p by p positive definite symmetric R and

$${}_0F_1(b; Z) = \sum_{k=0}^{\infty} \sum_{\kappa \in \mathscr{P}_k} 1/(k!(b)_\kappa)\, C_\kappa(Z).$$

Hayakawa [4] shows that $H_\kappa(T) = (-1)^k L_\kappa^{(n/2)-p}(TT')$.

THEOREM 6. *Let* $|x| < 1$, $\kappa \in \mathscr{P}_k$, *and let* Z *be a* $p \times p$ *symmetric matrix. Then*

$$\sum_{l=k}^{\infty} \sum_{\lambda \in \mathscr{P}_l} \binom{\lambda}{\kappa} x^l L_\lambda^\gamma(Z)/l!$$

$$= (1-x)^{-(1/2)pq} \operatorname{etr}(-x(1-x)^{-1}Z)[x(1-x)^{-1}]^k L_\kappa^\gamma((1-x)^{-1}Z)/k!. \tag{4.10}$$

Proof. This follows by making the change of variables $S = (1-x)R$ in the integral obtained by substituting (4.9) in the l.h.s. of (4.10) and applying Theorem 1. ∎

One can derive corollaries to Theorems 4–6 that are analogous to the corollary to Theorem 3, using essentially the same proof.

APPENDIX: $\binom{\lambda}{P}$ FOR ALL MONOMIALS P IN s_j OF DEGREES 4–5

In Table 2 are selected values for $f_j(\kappa)$ and $Q_j(\kappa)$ for $\kappa \in \mathscr{P}_r$. These were used together with Table 1, Eqs. (3.12)–(3.15) and the relations between the C_λ and the monomials in s_j (e.g., [3]) to derive the following expressions. These are not always expressed as compactly as possible, but the form has been chosen to standardize the form of the terms involved.

$$\binom{\lambda}{s_4} = (1/12)\left[2d_4(\lambda) - 3d_3(\lambda) - 2(l-2)\,d_2(\lambda) + \binom{l}{2}\right],$$

$$\binom{\lambda}{s_3 s_1} = (1/16)\left[2(l-3)\,d_3(\lambda) - 3\binom{l}{3} - (l-2)\,d_2(\lambda) + d_2(\lambda) + \binom{l}{2}\right],$$

$$\binom{\lambda}{s_2^2} = (1/6)\left[d_2^2(\lambda) - 3d_3(\lambda) + d_2(\lambda) + \binom{l}{2}\right],$$

$$\binom{\lambda}{s_2 s_1^2} = (1/12)(l-3)(l-2)\,d_2(\lambda),$$

$$\binom{\lambda}{s_1^4} = \binom{l}{4},$$

$$\begin{aligned}\binom{\lambda}{s_5} = (1/96)\Big[&4d_5(\lambda) - 12d_4(\lambda) - 4d_2^2(\lambda) - 6(l-3)\,d_3(\lambda) + 13d_3(\lambda)\\ &+ 9(l-2)\,d_2(\lambda) + 5\binom{l}{3} - d_2(\lambda) - 6\binom{l}{2}\Big],\end{aligned}$$

$$\begin{aligned}\binom{\lambda}{s_4 s_1} = (1/60)\Big[&2(l-4)\,d_4(\lambda) - 2(l-3)(l-2)\,d_2(\lambda) - 3(l-3)\,d_3(\lambda)\\ &+ 3d_3(\lambda) + 2(l-2)\,d_2(\lambda) + 3\binom{l}{3} - 2\binom{l}{2}\Big],\end{aligned}$$

$$\begin{aligned}\binom{\lambda}{s_3 s_2} = (1/80)\Big[&4d_3(\lambda)\,d_2(\lambda) - 16d_4(\lambda) - 2d_2^2(\lambda) - (l-3)(l-2)\,d_2(\lambda)\\ &+ 18d_3(\lambda) + 8(l-2)\,d_2(\lambda) + d_2(\lambda) - 5\binom{l}{2}\Big],\end{aligned}$$

$$\begin{aligned}\binom{\lambda}{s_3 s_1^2} = (1/80)\Big[&2(l-4)(l-3)\,d_3(\lambda) - (l-3)(l-2)\,d_2(\lambda) - 12\binom{l}{4}\\ &+ 2(l-2)\,d_2(\lambda) + 6\binom{l}{3} - 2d_2(\lambda) - 2\binom{l}{2}\Big],\end{aligned}$$

$$\begin{aligned}\binom{\lambda}{s_2^2 s_1} = (1/30)\Big[&(l-4)\,d_2^2(\lambda) - 3(l-3)\,d_3(\lambda) + 3d_3(\lambda) + 3\binom{l}{3}\\ &+ (l-2)\,d_2(\lambda) - 2d_2(\lambda) - 2\binom{l}{2}\Big],\end{aligned}$$

$$\binom{\lambda}{s_2 s_1^3} = (1/60)(l-4)(l-3)(l-2)\,d_2(\lambda),$$

$$\binom{\lambda}{s_1^5} = \binom{l}{5}.$$

TABLE 2

Coefficients $f_j = f_j(\kappa)$ and $Q_j = Q_j(\kappa)$ as Defined by (3.10)

κ	f_0, Q_0	$2f_1, -2Q_1$	$4f_2$	$8f_3$	$4Q_2$	$8Q_3$
(2)	1	2	0	0	4	-8
(1^2)	1	-1	0	0	1	1
(3)	1	6	4	0	28	-120
(21)	1	1	-2	0	3	-5
(1^3)	1	-3	2	0	7	15
(4)	1	12	44	48	100	-720
(31)	1	5	2	-8	23	-97
(2^2)	1	2	-1	-2	5	-10
(21^2)	1	-1	-4	4	5	5
(1^4)	1	-6	11	-6	25	90

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